www.ietdl.org

Published in IET Control Theory & Applications
Received on 14th April 2014
Revised on 22nd August 2014
Accepted on 17th September 2014
doi: 10.1049/iet-cta.2014.0642

ISSN 1751-8644

# Non-linear Mittag–Leffler stabilisation of commensurate fractional-order non-linear systems

*Dongsheng Ding, Donglian Qi, Qiao Wang*

*College of Electrical Engineering, Zhejiang University, Hangzhou 310027, People's Republic of China*
*E-mail: donsding@126.com*

**Abstract:** Mittag–Leffler stability is a property of fractional-order dynamical systems, also called fractional Lyapunov stability, requiring the evolution of the positive-definite functions to be Mittag–Leffler, rather than the exponential meaning in Lyapunov stability theory. Similarly, fractional Lyapunov function plays an important role in the study of Mittag–Leffler stability. The aim of this study is to create closed-loop systems for commensurate fractional-order non-linear systems (FONSs) with Mittag–Leffler stability. We extend the classical backstepping to fractional-order backstepping for stabilising (uncertain) FONSs. For this purpose, several conditions of control fractional Lyapunov functions for FONSs are investigated in terms of Mittag–Leffler stability. Within this framework, (uncertain) FONSs Mittag–Leffler stabilisation is solved via fractional-order backstepping and the global convergence of closed-loop systems is guaranteed. Finally, the efficiency and applicability of the proposed fractional-order backstepping are demonstrated in several examples.

## 1 Introduction

Recently, fractional-order systems (FOSs) have been widely studied in engineering and mathematical sciences [1–3]. This is mainly due to the fact that many physical phenomena are well characterised by fractional-order differential equations [3], such as hereditary and non-locality, self-similarity and stochaticity. Modelling of transmission lines, viscoelastic materials, cell diffusion processes, complex behaviours of polymers, electromagnetic waves and electrode–electrolyte polarisation are some typical applications. For more details on the applications of fractional calculus, one can refer to the monographs [1–4], the papers [5–14] and references therein.

Owing to the broad applications of FOSs, FOS stabilisation problem has become an interesting topic and has attracted a lot of attentions among researchers and scientists in recent years. Most of the known results on FOSs stabilisation concentrated on fractional-order linear systems (FOLSs). The early stability criterion of FOLS is the Matignon theorem [15]. And its linear matrix inequality (LMI) representations were proposed by Sabatier *et al.* [4] and the sufficient and necessary conditions were investigated by [16–18] further. With respect to LMI conditions, the pseudo-state feedback stabilisation of deterministic FOLSs was addressed in [19, 20]. On the other hand, many new robust stabilisation results were put forward in [5–9] recently. For early robust stabilisation results one can refer to [16–18] and references in [5–9]. Besides, $\mathcal{H}_\infty$ control problems of FOSs were proposed by authors in [7–9]. However, real FOSs always have many non-linear structures.

So far, fractional-order non-linear systems (FONSs) have formed a new class of non-linear systems [1, 2]. It is well known that the Lyapunov direct method is the fundamental tool to stabilise non-linear systems. Recently, Lyapunov-like stability of FOSs has been discussed in several papers. An early Lyapunov-like theory for FOSs was investigated by Lakshmikantham *et al.* [21]. In [22, 23], Mittag–Leffler stability and generalised Mittag–Leffler stability were introduced to describe Lyapunov-like stability of FOSs. Generalised Mittag–Leffler stability of multi-variables FOSs was investigated by Yu *et al.* [24] further. Although there is no difficulty to apply Lyapunov-like technique to FOSs in [25], the parametric stability condition seems too parsimonious. Moreover, some Lyapunov functionals were proposed by Burton [26] to prove the stability of FOSs. From different perspectives, Wang *et al.* [27] introduced Hyers–Ulam–Rassias stability and Hyers–Ulam stability for FOSs. However, finding appropriate Lyapunov-like functions for FOSs remains a tedious task. Some existing possible Lyapunov-like functions for FOSs can be found in [25, 28, 29].

An attention should be paid to, Mittag–Leffler stability just describes the pseudo-state trajectories, not real states of FOSs, which you can refer to [19, 25, 30, 31] for distinguishing them. So, we call the Lyapunov function that are constructed for Mittag–Leffler stability as fractional Lyapunov function and this technique as the fractional Lyapunov direct method.

To the best of authors' knowledge, few results on FONSs stabilisation have been reported in terms of Mittag–Leffler stability. In [10], linear state feedback was introduced to stabilise linearised FONSs using the eigenvalue analysis. Robust stabilisation of fractional-order non-linear complex networks was investigated by Lan *et al.* [11] via the Lyapunov indirect approach. For some simple examples of FONSs stabilisation, one can refer to [10, 21–27, 29]. Recently, fractional-order sliding-mode control is well defined for stabilising some specific FONSs, which you can refer to [12–14, 32] and references therein. So far, the stabilisation problem of FONSs remains an open problem, especially in terms of Mittag–Leffler stability.

Motivated by the mentioned developments, we concentrate on non-linear Mittag–Leffler stabilisation of commensurate FONSs. As we know, backstepping is a well-known efficient methodology of stabilising non-linear systems, which has been widely applied in practical applications [33]. However, to the best of authors' knowledge, backstepping is restricted to the classical integer-order non-linear systems. There are few results on it, besides the first example proposed in [1]. Therefore there are many works to do with backstepping control laws design for FONSs. As the resulting control laws are with fractional-order forms, we call such methodology the fractional-order backstepping.

In our contributions, the Mittag–Leffler stabilisation problem of commensurate FONSs is solved with a guaranteed global convergence of closed-loop systems. Firstly, a general framework of fractional Lyapunov function-based design is well defined via control fractional Lyapunov function (cflf) for FOSs. Several conditions of cflfs for FONSs are investigated in terms of Mittag–Leffler stability. Within this framework, fractional-order backstepping is shown by extending the classical backstepping. The analytic form of (adaptive) feedback control laws of stabilising deterministic or uncertain fractional-order non-linear systems (DFONSs or UFONSs) are designed via fractional-order backstepping. Several examples demonstrate the efficiency of the proposed fractional-order backstepping. These developments provide a systematic method of constructing Mittag–Leffler stable closed-loop systems for FONSs and a global convergence is built into them.

The rest of the paper is organised as follows. Some definitions are introduced in Section 2 and a class of cflfs is defined. Main results of fractional-order backstepping are presented in Section 3. Section 3.1 deals with the non-linear Mittag–Leffler stabilisation of DFONSs. Moreover, the additional adaptive laws will be included to adaptively stabilise FONSs with unknown constant parameters in Section 3.2. The efficiency and applicability of fractional-order backstepping are verified in simulations in Section 4. Finally, some conclusions are summarised in Section 5.

## 2 Preliminaries

In this section, some definitions of fractional calculus are introduced. Our main result is to propose a general framework of fractional Lyapunov function-based design via cflf for FOSs. A class of cflfs is defined for FOSs and the sufficient conditions of cflfs are provided.

### 2.1 Fractional-order calculus

*Definition 1 (Caputo fractional-order derivative [3, 34]):* Let $f(t)$ is a real continuously differentiable function. The Caputo fractional-order derivative with order $0 < \nu < 1$ on $t > 0$ is defined by

$$D_t^\nu f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu-n+1}} \, \mathrm{d}\tau \tag{1}$$

where $n = \lceil \nu \rceil$, $\nu > 0$, $\lceil \cdot \rceil$ is the ceiling function and the initial time is 0.

For simplicity, the symbol $D^\nu$ is shorted for $D_t^\nu$. The fractional-order derivative of a constant $C$ is 0. A special attention should be paid to the property of 'No violation of the Leibniz rule. No fractional derivative' [35]. The Leibniz rule for fractional-order derivative is a infinite sum [34], which cannot be used for fractional Laypunov-based analysis.

Another setback of fractional calculus is about the chain rule for fractional-order derivative. Let a real continuously differentiable composite function $f \circ g(t)$, its fractional-order derivative is an infinite sum, which you can refer to [34].

To sum up, as the Leibniz rule and the chain rule can result in infinitely many terms, it is difficult for a class of common quadratic Lyapunov functions to figure out the Mittag–Leffler stability of FONSs [1]. This phenomena is the main reason for the difficulty of fractional Lyapunov based design to stabilise FONSs. It seems that the fractional Lyapunov direct method is invalid for FONSs Mittag–Leffler stability analysis. Fortunately, within the framework of Mittag–Leffler stability [22–24], we can start from $V = \frac{1}{2}x^2$ to generalise a class of fractional Lyapunov functions.

### 2.2 Fractional-order extension of control Lyapunov function

Firstly, by the use of Mittag–Leffler stability, we give a general framework of fractional Lyapunov function-based design via cflf for FOSs. Then the power law for fractional-order derivative is presented. A class of possible fractional Lyapunov functions is investigated and the conditions of cflfs are defined for FONSs Mittag–Leffler stabilisation.

*Theorem 1 (Mittag–Leffler stability (Fractional Lyapunov stability) [23]):* Let $x(t) = 0$ be the equilibrium point of the FOS $D^\nu x = f(x, t)$, $x \in \Omega$, where $\Omega$ is a neighbourhood region of the origin. Assume that there exists a fractional Lyapunov function $V(t, x(t)) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and $K$-class functions $\gamma_i$, $i = 1, 2, 3$ satisfying

(*i*) $\gamma_1(\|x\|) \le V(t, x(t)) \le \gamma_2(\|x\|)$;
(*ii*) $D^\nu V(t, x(t)) \le -\gamma_3(\|x\|)$.

Then the FOS is asymptotically Mittag–Leffler stable. Moreover, if $\Omega = \mathbb{R}^n$, the FOS is globally asymptotically Mittag–Leffler stable.

Now by the use of Theorem 1, the concept of control fractional Lyapunov function (cflf) is extended to test whether a FOS is feedback Mittag–Leffler stabilisable by applying the control law $u$.

*Definition 2 (cflf for FOSs):* A smooth function $V(t, x(t)) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is called a cflf for the FOS $D^\nu x = f(x, u)$, $x \in \mathbb{R}^n$, $f(0, 0) = 0$ with the control law $u = \alpha(x)$ if there exist three $K$-class functions $\gamma_i$, $i = 1, 2, 3$, such that

(*i*) $\gamma_1(\|x\|) \le V(t, x(t)) \le \gamma_2(\|x\|)$;
(*ii*) $D^\nu V(t, x(t)) \le -\gamma_3(\|x\|)$.

In the case of designing adaptive control laws for UFONSs, the adaptive parameters may appear in $V$, the cflf is called adaptive control fractional Lyapunov function (acflf). The aim of FONSs stabilisation is to design a feedback control law $u = \alpha(x)$ such that the equilibrium




doi: 10.1049/iet-cta.2014.0642

$x = 0$ of the closed-loop system $D^\nu x = f(x, \alpha(x))$ is (globally) asymptotically Mittag–Leffler stable. Actually, finding $\alpha(x)$ and $V(t, x(t))$ satisfying *(i)* and *(ii)* in Definition 2 is a difficult task in most cases. The symbol $\|\cdot\|$ represents Euclidean norm in the controller design.

Finding fractional Lyapunov functions for FOSs have been investigated in [21–29]. To include the fractional-order derivative trajectories information for constructing feedback control laws effectively, we propose a general class of cflfs and its sufficient conditions are given in Lemmas 2–4. Firstly, we give the power law for fractional-order derivative.

*Lemma 1 (Power law for fractional-order derivative):* Let $x(t) \in \mathbb{R}$ be a real continuously differentiable function. Then for any $p = 2^n$, $n \in \mathbb{N}$,

$$D^\nu x^p(t) \le p x^{(p-1)}(t) D^\nu x(t) \tag{2}$$

where $0 < \nu < 1$ is the fractional order.

*Proof:* The case $p = 2$ is established firstly. Consider $2x(t)D^\nu x(t) - D^\nu x^2(t)$, from Lemma 1 in [29], we have $\frac{1}{2} D^\nu x^2(t) \le x(t) D^\nu x(t)$.

Then, by the use of recursion, for any $p = 2^n$, $n \in \mathbb{N}$, we have

$$\begin{aligned} D^\nu x^p(t) &\le 2x^{p/2}(t) D^\nu x^{p/2}(t) \\ &\le 2^2 x^{(p/2 + p/2^2)}(t) D^\nu x^{p/2^2}(t) \\ &\le \cdots \le p x^{(p-1)}(t) D^\nu x(t) \end{aligned}$$

Thus the inequality (2) is proved completely. □

*Remark 1 (Power law for integer-order derivative):* It is obvious that the classical chain rule becomes invalid for Caputo fractional-order derivative. Only when $\nu = 1$, the inequality (2) is reduced to the equality $D^1 x^p(t) = p x^{p-1}(t) x'(t)$, which is the same as the classical power law.

Next, we consider the conditions of cflf for FOSs. Using a general positive-definite function $V = \frac{1}{p} x^p$, $p = 2^n$, $n \in \mathbb{N}$ as the fractional Lyapunov function, we aim to construct the conditions that making it a reasonable cflf for a scalar FOS. The sufficient conditions can be expressed as the followings.

*Lemma 2 (A special cflf for FOSs):* For the FOS $D^\nu x = f(x, u)$, $x \in \mathbb{R}$, $0 < \nu < 1$, $f(0, 0) = 0$ with the control law $u = \alpha(x)$ is asymptotically Mittag–Leffler stable if for $p = 2^n$, $n \in \mathbb{N}$, there exists a $K$-class functions $\gamma$, such that

$$x^{p-1} D^\nu x = x^{p-1} f(x, \alpha(x)) \le -\gamma(\|x\|) \tag{3}$$

*Proof:* Let the candidate cflf $V = \frac{1}{p} x^p$, it is obvious by the use of Lemma 1 and Definition 2. □

Another two versions of Lemma 2 are described as the following Lemmas 3 and 4.

*Lemma 3 (Conservative version of Lemma 2):* For the FOS $D^\nu x = f(x, u)$, $x \in \mathbb{R}$, $0 < \nu < 1$, $f(0, 0) = 0$ with the control law $u = \alpha(x)$ is stable if for a $p = 2^n$, $n \in \mathbb{N}$,

$$x^{p-1} D^\nu x = x^{p-1} f(x, \alpha(x)) \le 0 \tag{4}$$

And the system with $u = \alpha(x)$ is asymptotically Mittag–Leffler stable if $x^{p-1} f(x, \alpha(x)) < 0$.

*Proof:* Let the candidate cflf $V = \frac{1}{p} x^p$, with respect to Lemma 2, we have $D^\nu V \le x^{p-1}(t) D^\nu x(t)$.

When $x^{p-1} f(x, \alpha(x)) \le 0$, by the use of fractional-order comparison principle [22], we have $V \le V(x(0))$, $x \in \mathbb{R}$. It implies that the system is stable in terms of Mittag–Leffler stability.

When $x^{p-1} f(x, \alpha(x)) < 0$, then exists a $K$-class function $\gamma$ such that $D^\nu V \le -\gamma(\|x\|)$. Therefore it is obvious that the system is asymptotically Mittag–Leffler stable by the use of Theorem 1. □

When $p = 2$, the proof of Lemma 3 was given in [29]. Besides, the vector form of the general positive function $V = \frac{1}{2} x^{\mathrm{T}} P x$, $x \in \mathbb{R}^n$, $P = \mathrm{diag}[p_1, \ldots, p_n] > 0$ can be chosen as a cflf for a vector FOS.

*Lemma 4 (Vector version of Lemma 2):* For the FOS $D^\nu x = f(x, u)$, $x \in \mathbb{R}^n$, $0 < \nu < 1$, $f(0, 0) = 0$ with the control law $u = \alpha(x)$ is asymptotically Mittag–Leffler stable if for $P = \mathrm{diag}[p_1, \ldots, p_n] > 0$, there exists a $K$-class functions $\gamma$, such that

$$x^{\mathrm{T}} P D^\nu x = x^{\mathrm{T}} P f(x, \alpha(x)) \le -\gamma(\|x\|)$$

*Proof:* Let the candidate cflf $V = \frac{1}{2} x^{\mathrm{T}} P x$, the proof is similar to Lemma 2. □

*Remark 2 (A simple cflf):* For the case $p = 2$, choose the cflf $V = \frac{1}{2} x^2$. By use of Lemma 3, the FOS $D^\nu x = f(x, u)$, $x \in \mathbb{R}$, $f(0, 0) = 0$ with the control law $u = \alpha(x)$ is stable if $x D^\nu x = x f(x, \alpha(x)) \le 0$. The system with $u = \alpha(x)$ is asymptotically Mittag–Leffler stable if $x f(x, \alpha(x)) < 0$. Actually, this cflf is valid for the classical integer-order backstepping design [33].

*Remark 3 (Conservative in a certain degree):* Two aspects of the conservative in the conditions of Lemmas 2–4 should be noted. The above conditions based on the sufficient conditions of Theorem 1 are sufficient. It is possible that there exist other better candidate cflfs, which may contradict with Theorem 1. On the other hand, the candidate cflfs used in Lemmas 2–4 are only a class of possible cflfs. However, if other more complex cflfs may be chosen, due to the facts in Section 2.1, the complexity of control laws design will increase apparently.

## 3 Main results

Two kinds of non-linear Mittag–Leffler stabilisation problems of commensurate FONSs are considered. The non-linear Mittag–Leffler stabilisation of DFONSs is introduced in Section 3.1 via fractional-order integrator backstepping and a recursive design is provided. However, a common class of FONSs can be described by fractional-order non-linear models with unknown constant parameters, that is, UFONSs. The additional adaptive laws will be included to adaptively stabilise such FONSs in Section 3.2.

### 3.1 Non-linear Mittag–Leffler stabilisation of DFONS

To introduce fractional-order integrator backstepping for FONSs, the following assumption is given.

*Assumption 1:* Let the DFONS $D^{\nu}x = f(x) + g(x)u, f(0) = 0$, where $x \in \mathbb{R}^n$ is the pseudo-state and $u \in \mathbb{R}$ is the control input. There exists a continuously differentiable feedback control law $u = \alpha(x)$, $\alpha(0) = 0$ and a $K$-class function $\gamma$ such that

$$x^{\mathrm{T}} D^{\nu} x = x^{\mathrm{T}}[f(x) + g(x)\alpha(x)] \le -\gamma(\|x\|), \quad x \in \mathbb{R}^n \qquad (5)$$

*Remark 4:* In this case, $V = \frac{1}{2}x^{\mathrm{T}}x$ is a candidate cflf according to Definition 2. For the scalar case $x \in \mathbb{R}$, Assumption 1 is always valid for $g(x) \neq 0$. The control $u$ can be set to $\alpha(x) = \frac{-1}{g(x)}[f(x) + Cx]$, and $D^{\nu}V(x) \le -Cx^2$, where $C > 0$ is an constant.

A special attention should be paid to all possible control laws. Although the inequality (5) provides a wide variety of control laws, the conservative of the specific cflf is unavoidable. However, the efficiency of (5) is obvious for common FONSs, which will be justified later.

*Theorem 2:* Let the DFONS,

$$\begin{cases} D^{\nu}x = f(x) + g(x)\xi \\ D^{\nu}\xi = u \end{cases} \qquad (6)$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}$ are the pseudo-states and $u \in \mathbb{R}$ is the control input. Suppose $D^{\nu}x = f(x) + g(x)\xi$ satisfies Assumption 1 with $\xi$ as its virtual control. Let the cflf $V_a(x, z) = \frac{1}{2}x^{\mathrm{T}}x + \frac{1}{2}z^2$, $z = \xi - \alpha(x)$, that is, there exists a feedback control $u = \alpha_a(x, \xi)$ which renders $(0, 0)$ the globally asymptotically Mittag–Leffler stable equilibrium. One such control is

$$u = -C[\xi - \alpha(x)] - x^{\mathrm{T}}g(x) + D^{\nu}\alpha(x) \qquad (7)$$

where $C > 0$ is a constant.

*Proof:* Introduce the error $z = \xi - \alpha(x)$, and take fractional-order derivative, we have

$$D^{\nu}x = f(x) + g(x)[z + \alpha(x)], D^{\nu}z = u - D^{\nu}\alpha(x)$$

By use of the cflf $V_a(x, \xi)$, we have

$$\begin{aligned} D^{\nu}V_a(x, z) &\le x^{\mathrm{T}}[f(x) + g(x)\alpha(x)] \\ &\quad + z[u + x^{\mathrm{T}}g(x) - D^{\nu}\alpha(x)] \\ &\le -\gamma(\|x\|) + z[u + x^{\mathrm{T}}g(x) - D^{\nu}\alpha(x)] \end{aligned}$$

where the terms containing $z$ as a factor have been grouped together.

By use of the Assumption 1, any control $u$ which renders $D^{\nu}V_a(x, \xi) \le -\gamma(\|x\|) - Cz^2$ can be chosen as (7). By use of Euclidean norm, there exists a $K$-class function $\bar{\gamma}(\|\bar{x}\|) = \gamma(\|x\|) + Cz^2$, $\bar{x} = [x^{\mathrm{T}}, z]^{\mathrm{T}}$.

With respect to Lemma 2, the cflf holds globally. So far, this proof is completed. □

*Example 1:* Consider a fractional-order non-linear planar system

$$\begin{cases} D^{\nu}x = x\xi \\ D^{\nu}\xi = u \end{cases} \qquad (8)$$

It is obvious that $f(x) = 0$, $g(x) = x$. Let $V = \frac{1}{2}x^2$, $z = \xi - \alpha(x)$, we have

$$D^{\nu}V(x) \le xD^{\nu}x = x^2[z + \alpha(x)]$$

If choose $\alpha(x) = -Cx^2$, $C > 0$, then the FOS becomes

$$\begin{cases} D^{\nu}x = xz - Cx^3 \\ D^{\nu}z = u - D^{\nu}\alpha(x) \end{cases}$$

Let the candidate cflf $V_a(x, \xi) = V(x) + \frac{1}{2}[\xi - \alpha(x)]^2$, we have

$$D^{\nu}V_a(x, \xi) \le -Cx^4 + z[u + x^2 - D^{\nu}\alpha(x)]$$

By use of Theorem 2, the control can be chosen by

$$u = -C_1[\xi - \alpha(x)] - x^2 + D^{\nu}\alpha(x)$$

where $C$, $C_1 > 0$ are constants.

Theorem 2 and Example 1 show how to add a single fractional-order integrator. The following theorem can be repeatedly applied to add a whole chain of fractional-order integrators.

*Theorem 3:* Let the DFONS is described by the chain of fractional-order integrators

$$\begin{cases} D^{\nu}x = f(x) + g(x)\xi_1 \\ D^{\nu}\xi_1 = \xi_2 \\ \dots \\ D^{\nu}\xi_{n-1} = \xi_n \\ D^{\nu}\xi_n = u \end{cases} \qquad (9)$$

where $x \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$, $i = 1, \dots, n$ are the pseudo-states and $u \in \mathbb{R}$ is the control input. Let $D^{\nu}x = f(x) + g(x)\xi_1$ satisfies Assumption 1 with $\xi_1$ as its virtual control, and $\alpha(x) = \alpha_0(x)$. If the cflf is taken by

$$V_a(x, \xi) = \frac{1}{2}x^{\mathrm{T}}x + \frac{1}{2}\sum_{i=1}^{n} z_i^2 \qquad (10)$$

that is, there exists a feedback control u which renders (0,…,0) the globally asymptotically Mittag–Leffler stable.

One such control is

$$\begin{aligned} u &= -C_n z_n - z_{n-1} + D^{\nu}\alpha_{n-1} \\ \alpha_{i-1} &= -C_{i-1}z_{i-1} - z_{i-2} + D^{\nu}\alpha_{i-2}, i = 2, 3, \dots, n \\ \xi_i &= z_i + \alpha_{i-1}(x, \xi_1, \dots, \xi_{i-1}), i = 1, 2, \dots, n \end{aligned} \qquad (11)$$

where $C_1, \dots, C_n > 0$ are constants.

*Proof:* By repeating Theorem 2 with $\xi_1, \dots, \xi_n$ as virtual controls, we have the following steps.

*Step 1.* Let $z_1 = \xi_1 - \alpha_0(x, \xi_1)$, $z_2 = \xi_2 - \alpha_1(x, \xi_1)$, the first fractional Lyapunov function $V_1(x, z_1) = \frac{1}{2}x^{\mathrm{T}}x + \frac{1}{2}z_1^2$ and view $\xi_1$ as a virtual control. With Assumption 1, we have

$$\begin{aligned} D^{\nu}V_1 &\le x^{\mathrm{T}}[f(x) + g(x)\alpha_0] + z_1[x^{\mathrm{T}}g(x) + z_2 + \alpha_1 - D^{\nu}\alpha_0] \\ &\le -\gamma(\|x\|) + z_1[x^{\mathrm{T}}g(x) + z_2 + \alpha_1 - D^{\nu}\alpha_0] \end{aligned}$$

If choose $\alpha_1(x, \xi_1) = -C_1 z_1 - x^{\mathrm{T}}g(x) + D^{\nu}\alpha_0$, $z_2$ is to be governed to zero.



IET Control Theory Appl., 2015, Vol. 9, Iss. 5, pp. 681–690
doi: 10.1049/iet-cta.2014.0642

*Step 2*. Let $z_3 = \xi_3 - \alpha_2(x, z_1, z_2)$, the second fractional Lyapunov function $V_2(x, z_1, z_2) = V_1 + \frac{1}{2}z_2^2$ and view $\xi_2$ as a virtual control, we have

$$\begin{aligned} D^\nu V_2 &\le -\gamma(\|x\|) - C_1 z_1^2 + z_1 z_2 + z_2 D^\nu z_2 \\ &= -\gamma(\|x\|) - C_1 z_1^2 + z_2[z_3 + z_1 + \gamma_2 - D^\nu \gamma_1] \end{aligned}$$

If choose $\gamma_2(x, z_1, z_2) = -C_2 z_2 - z_1 + D^\nu \alpha_1$, $z_3$ is to be governed to zero.
*Step 3*. Let $z_4 = \xi_4 - \alpha_3(x, z_1, z_2, z_3)$, the third fractional Lyapunov function $V_3(x, z_1, z_2, z_3) = V_2 + \frac{1}{2}z_3^2$ and view $\xi_3$ as a virtual control, we have

$$\begin{aligned} D^\nu V_3 &\le -\gamma(\|x\|) - C_1 z_1^2 - C_2 z_2^2 + z_2 z_3 + z_3 D^\nu z_3 \\ &= -\gamma(\|x\|) - C_1 z_1^2 - C_2 z_2^2 + z_3[z_4 + z_2 + \alpha_3 - D^\nu \alpha_2] \end{aligned}$$

If choose $\alpha_3(x, \xi_1, \xi_2, \xi_3) = -C_3 z_3 - z_2 + D^\nu \alpha_2$, $z_4$ is to be governed to zero.
*Step n − 1*. Let $z_n = \xi_n - \alpha_{n-1}(x, z_1, \ldots, z_{n-1})$, the $(n-1)$th fractional Lyapunov function $V_{n-1}(x, z_1, \ldots, z_{n-1}) = V_{n-2} + \frac{1}{2}z_{n-1}^2$ and view $\xi_{n-1}$ as a virtual control, we have

$$\begin{aligned} D^\nu V_{n-1} &\le -\gamma(\|x\|) - \sum_{i=1}^{n-2} C_i z_i^2 + z_{n-2} z_{n-1} + z_{n-1} D^\nu z_{n-1} \\ &= -\gamma(\|x\|) - \sum_{i=1}^{n-2} C_i z_i^2 + z_{n-1}[z_n + z_{n-2} + \alpha_{n-1}) \\ &\quad + D^\nu \alpha_{n-2}] \end{aligned}$$

If choose $\alpha_{n-1} = -C_{n-1} z_{n-1} - z_{n-2} + D^\nu \alpha_{n-2}$, $z_n$ is to be governed to zero. The cflf and one control can be chosen by (10) and (11), respectively. With Assumption 1, we have

$$D^\nu V_a(x, z_1, \ldots, z_n) \le -\gamma(\|x\|) - \sum_{i=1}^{n} C_i z_i^2$$

By use of Euclidean norm, there exists a $K$-class function

$$\bar{\gamma}(\|\bar{x}\|) = \gamma(\|x\|) + \sum_{i=1}^{n} C_i z_i^2, \bar{x} = [x^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}$$

With respect to Lemma 2, the cflf (10) holds globally. So far, the proof is completed. □

Theorems 2 and 3 show the recursive methodology of fractional-order backstepping, which may result fractional-order feedback control laws.

*Example 2:* Consider a fractional-order non-linear planar system

$$\begin{cases} D^\nu x = x\xi + x\Delta \\ D^\nu \xi = u \end{cases} \tag{12}$$

where $x, \xi \in \mathbb{R}$ are the pseudo-states and $u \in \mathbb{R}$ is the control input. $\Delta$ is the unknown bounded constant, but we do not known its bound.

It is obvious that $f(x) = x^2\Delta$, $g(x) = x$. The static feedback control law of Theorem 3 is considered here. Let $V = \frac{1}{2}x^2$, $z = \xi - \alpha(x)$, we have

$$D^\nu V(x) \le x D^\nu x = x^2[z + \alpha(x) + \Delta]$$

Choose $\alpha(x) = -Cx^2$, $C > 0$, the FONS becomes

$$\begin{cases} D^\nu x = xz - Cx^3 + x\Delta \\ D^\nu z = u - D^\nu \alpha(x) \end{cases}$$

The candidate cflf is $V_a(x, \xi) = V(x) + \frac{1}{2}[\xi - \alpha(x)]^2$. So, we have

$$D^\nu V_a(x, \xi) \le -Cx^4 + z[u + x^2 - D^\nu \alpha(x)] + x\Delta$$

By the use of Theorem 2, the control can be chosen as

$$\begin{aligned} u &= -C_1[\xi - \alpha(x)] - x^2 + D^\nu \alpha(x) \\ D^\nu V_a(x, \xi) &\le -Cx^4 - C_1 z^2 + x^2 \Delta \end{aligned}$$

where $C, C_1 > 0$ are constants. It is obvious that the global boundedness can be guaranteed by choosing $C > \|\Delta\|_\infty$.

In Example 2, although the static controller can guarantee that in the presence of bounded uncertainties the closed-loop pseudo-states remain bounded, the feedback gain may increase too large (high gain feedback). If the uncertain parameters are unknown, this method may be invalid. In the next section, fractional-order adaptive control laws will be included to deal with this case.

### 3.2 Non-linear Mittag–Leffler stabilisation of UFONS

A common form of non-linearities appears multiplied with physical constants, often poorly known or dependent on the slowly changing environment [33]. We consider the unknown constant parameters appear linearly in the fractional-order models. In presence of such parametric uncertainties, the adaptive fractional-order backstepping is introduced to achieve convergence of the closed-loop system.

To introduce fractional-order backstepping for UFONSs, the following assumption is given.

*Assumption 2:* Let the UFONS $D^\nu x = f(x) + F(x)\theta + g(x)u$, where $x \in \mathbb{R}^n$ is the pseudo-state, $\theta \in \mathbb{R}^m$ is an unknown constant parameter and $u = \alpha(x, \hat{\theta}) \in \mathbb{R}$ is the control input. There exists a adaptive feedback control law $u = \alpha(x, \hat{\theta})$ and a $K$-class function $\gamma$ such that

$$x^{\mathrm{T}}[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})] \le -\gamma(\|\bar{x}\|) \tag{13}$$

$$D^\nu \hat{\theta} = \Gamma F(x)^{\mathrm{T}} x \tag{14}$$

where $\bar{x} = [x^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}}]^{\mathrm{T}}$, $\tilde{\theta} = \theta - \hat{\theta} \in \mathbb{R}^m$ is the parameter estimate error and $\Gamma = \mathrm{diag}[p_1, \ldots, p_m] > 0$ is the gain matrix of the adaptive law.

*Remark 5:* Let the candidate acflf $V = \frac{1}{2}x^{\mathrm{T}}x + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}$, with the adaptive control law $\alpha$ and (14), we have $D^\nu V(x) \le x^{\mathrm{T}}[f(x) + F(x)\hat{\theta} + g(x)\alpha(x, \hat{\theta})] + \tilde{\theta}^{\mathrm{T}}[F(x)^{\mathrm{T}}x - \Gamma^{-1}D^\nu \hat{\theta}]$. The sufficiency of Assumption 2 is obvious.

For the scalar case $x \in \mathbb{R}$, $\theta \in \mathbb{R}$, Assumption 2 is always valid for $g(x) \neq 0$, $x \in \mathbb{R}$. The adaptive control $u$ can be set to $\alpha(x, \hat{\theta}) = \frac{-1}{g(x)}[f(x) + F(x)\hat{\theta} + Cx]$, and $D^\nu \hat{\theta} = \Gamma F(x)x$, where $C > 0$ is a constant. Unless $x = 0$, we have $D^\nu V(x) < 0$. There exists a $K$-class function $\gamma$ such that $D^\nu V(x) \leq -\gamma(\|\bar{x}\|)$.

Similar to Assumption 1, a special attention should be paid to the conservative of possible control laws, which satisfies (13) and (14). However, the efficiency of (13) and (14) is obvious for common UFONSs, which will be justified later.

*Theorem 4:* Let the UFONS,

$$\begin{cases} D^\nu x = f(x) + F(x)\theta + g(x)\xi \\ D^\nu \xi = u \end{cases} \tag{15}$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}$ are the pseudo-states, $\theta \in \mathbb{R}^m$ is an unknown constant and $u \in \mathbb{R}$ is the control input. Let $D^\nu x = f(x) + F(x)\theta + g(x)\xi$ satisfies Assumption 2 with $\xi \in \mathbb{R}$ viewed as its virtual control. If the acflf is taken by

$$V_a(z_1, z_2, \hat{\theta}) = \frac{1}{2} z_1^{\mathrm{T}} z_1 + \frac{1}{2} z_2^2 + \frac{1}{2} \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \tilde{\theta} \tag{16}$$

where $z_1 = x$, $z_2 = \xi - \alpha(x, \hat{\theta})$ and $\tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimate error, that is, there exists an adaptive feedback control $u$ which renders the closed-loop system globally asymptotically Mittag–Leffler stable. The adaptive feedback control law can be chosen by

$$u = -x^{\mathrm{T}} g(x) - C_1[\xi - \alpha(x, \hat{\theta})] + D^\nu \alpha \tag{17}$$

$$D^\nu \hat{\theta} = \Gamma F(x)^{\mathrm{T}} x \tag{18}$$

where the adaptive parameter $\hat{\theta}$ is updated by (18), and $\Gamma = \mathrm{diag}[p_1, \ldots, p_m] > 0$ is the gain matrix of the adaptive law.

*Proof:* Two steps in this proof are presented as follows.
*Step 1*. Let $z_1 = x$ and $\xi$ viewed as the virtual control, the error $z_2 = \xi - \alpha(x, \hat{\theta})$, we have

$$D^\nu z_1 = f(z_1) + F(z_1)\theta + g(z_1)[z_2 + \alpha(x, \hat{\theta})].$$

Note $\tilde{\theta} = \theta - \hat{\theta}$, let the first fractional Lyapunov function $V_1(z_1, \hat{\theta}) = \frac{1}{2} z_1^{\mathrm{T}} z_1 + \frac{1}{2} \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \tilde{\theta}$. With Assumption 2, we have

$$\begin{aligned} D^\nu V_1 &\leq z_1^{\mathrm{T}}[f(z_1) + F(z_1)\hat{\theta} + g(z_1)\alpha(x, \hat{\theta})] + z_1^{\mathrm{T}} g(z_1) z_2 \\ &\quad + \tilde{\theta}[F(z_1)^{\mathrm{T}} z_1 - \Gamma^{-1} D^\nu \hat{\theta}] \\ &\leq -\gamma(\|\bar{x}\|) + z_1^{\mathrm{T}} g(z_1) z_2 + \tilde{\theta}[F(z_1)^{\mathrm{T}} z_1 - \Gamma^{-1} D^\nu \hat{\theta}] \end{aligned}$$

We postpone the choice of update law for $\hat{\theta}$ until the next step.
*Step 2*. To design the adaptive control $u$ for $D^\nu z_2 = u - D^\nu \alpha$. Consider the acflf (16), we have

$$\begin{aligned} D^\nu V_a &\leq -\gamma(\|\bar{x}\|) + z_1^{\mathrm{T}} g(z_1) z_2 \\ &\quad + \tilde{\theta}[F(z_1)^{\mathrm{T}} z_1 - \Gamma^{-1} D^\nu \hat{\theta}] + z_2[u - D^\nu \alpha] \\ &= -\gamma(\|\bar{x}\|) + \tilde{\theta}[F(z_1)^{\mathrm{T}} z_1 - \Gamma^{-1} D^n u \hat{\theta}] \\ &\quad + z_2[z_1^{\mathrm{T}} g(z_1) + u - D^\nu \alpha] \end{aligned}$$

One control and the adaptive law can be chosen by (17) and (18), respectively. So we have

$$D^\nu V_a \leq -\gamma(\|\bar{x}\|) - C_1 z_2^2$$

Similar to Theorem 2, the conditions in Lemma 2 are satisfied. The acflf (16) holds globally, so the proof is completed. □

The adaptive fractional-order backstepping is shown in Theorem 4 with respect to a single uncertain parameter. The following theorem concerns a general form.

*Theorem 5:* Let the parametric strict-feedback form of FONS

$$\begin{cases} D^\nu x_1 = x_2 + \varphi_1^{\mathrm{T}}(x_1)\theta \\ D^\nu x_2 = x_3 + \varphi_2^{\mathrm{T}}(x_1, x_2)\theta \\ \ldots \\ D^\nu x_{n-1} = x_n + \varphi_{n-1}^{\mathrm{T}}(x_1, \ldots, x_{n-1})\theta \\ D^\nu x_n = \beta(x) u + \varphi_n^{\mathrm{T}}(x)\theta \end{cases} \tag{19}$$

where $\beta(x) \neq 0$ for all $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$ is an unknown constant and $u \in \mathbb{R}$ is the control input. If the acflf is taken by

$$V_a(z_1, \ldots, z_k, \hat{\theta}) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 + \frac{1}{2} \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \tilde{\theta} \tag{20}$$

where $z_1 = x_1, z_i = x_i - \alpha_{i-1}(z_1, \ldots, z_{i-1}, \hat{\theta}), i = 2, \ldots, n$ and $\tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimate error, that is, there exists an adaptive feedback control $u$ which renders the closed-loop system globally asymptotically Mittag–Leffler stable on on the region $\Lambda$, where $\Lambda = \{(z_1, \ldots, z_n, \hat{\theta}) | z \neq 0\}$. And the boundedness of the closed-loop systems is guaranteed on $\mathbb{R}^{m+n} \backslash \Lambda$. The adaptive feedback control law can be chosen by

$$u = \frac{-1}{\beta(x)}[C_n z_n + z_{n-1} + \varphi_n^{\mathrm{T}}(x)\hat{\theta} - D^\nu \alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta})] \tag{21}$$

$$D^\nu \hat{\theta} = \Gamma \sum_{i=1}^{n} \varphi_i(x_1, \ldots, x_i) z_i \tag{22}$$

$$\begin{aligned} \alpha_{i-1}(z_1, \ldots, z_{i-1}, \hat{\theta}) = &-C_{i-1} z_{i-1} - z_{i-2} \\ &- \varphi_{i-1}^{\mathrm{T}}(x_1, \ldots, x_{i-1})\hat{\theta} \\ &+ D^\nu \alpha_{i-2}(z_1, \ldots, z_{i-2}, \hat{\theta}), \\ &\quad i = 3, \ldots, n \end{aligned}$$

where $\alpha_1(z_1, \hat{\theta}) = -C_1 z_1 - \varphi_1^{\mathrm{T}}(x_1)\hat{\theta}$ and $C_1, \ldots, C_n > 0$ are constants. The adaptive parameter $\hat{\theta}$ is updated by (22), and $\Gamma = \mathrm{diag}[p_1, \ldots, p_m] > 0$ is the gain matrix of the adaptive law.

*Proof:* By the use of recursion, we have the following steps.
*Step 1*. Let $z_1 = x_1$ and $x_2$ viewed as the virtual control, $z_2 = x_2 - \alpha_1(z_1, \hat{\theta})$, we have

$$D^\nu z_1 = z_2 + \alpha_1(z_1, \hat{\theta}) + \varphi_1^{\mathrm{T}}(x_1)\theta$$

Note $\tilde{\theta} = \theta - \hat{\theta}$, let the first fractional Lyapunov function $V_1(z_1, \hat{\theta}) = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \tilde{\theta}$, we have

$$\begin{aligned} D^\nu V_1 &\leq z_1[z_2 + \alpha_1(z_1, \hat{\theta}) + \varphi_1^{\mathrm{T}}(x_1)\hat{\theta}] \\ &\quad + \tilde{\theta}^{\mathrm{T}}(\varphi_1(x_1) z_1 - \Gamma^{-1} D^\nu \hat{\theta}) \end{aligned}$$

If choose $\alpha_1(z_1, \hat{\theta}) = -C_1 z_1 - \varphi_1^{\mathrm{T}}(x_1)\hat{\theta}$, $z_2, \tilde{\theta}$ are to be governed to zeros. Thus we have

$$D^\nu V_1(z_1, \hat{\theta}) \leq -C_1 z_1^2 + z_1 z_2 + \tilde{\theta}^{\mathrm{T}}(\varphi_1(x_1) z_1 - \Gamma^{-1} D^\nu \hat{\theta})$$



*IET Control Theory Appl.*, 2015, Vol. 9, Iss. 5, pp. 681–690
doi: 10.1049/iet-cta.2014.0642

*Step 2*. Let $z_3 = x_3 - \alpha_2(z_1, z_2, \hat{\theta})$, we have

$$D^{\nu} z_2 = z_3 + \alpha_2(z_1, z_2, \hat{\theta}) + \varphi_2^{\mathrm{T}}(x_1, x_2)\theta - D^{\nu}\alpha_1(z_1, \hat{\theta})$$

Let the second fractional Lyapunov function $V_2(z_1, z_2, \hat{\theta}) = V_1 + \frac{1}{2}z_2^2$, we have

$$\begin{aligned} D^{\nu} V_2 \le &-C_1 z_1^2 + z_1 z_2 + \hat{\theta}^{\mathrm{T}}(\varphi_1^{\mathrm{T}}(x_1) z_1 - \Gamma^{-1} D^{\nu}\hat{\theta}) \\ &+ z_2[z_3 + \alpha_2(z_1, z_2, \hat{\theta}) + \varphi_2^{\mathrm{T}}(x_1, x_2)\theta \\ &- D^{\nu}\alpha_1(z_1, \hat{\theta})] \\ = &-C_1 z_1^2 + z_1 z_2 + \tilde{\theta}^{\mathrm{T}} \\ &\times \left(\sum_{i=1}^{2} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^{\nu}\hat{\theta}\right) \\ &+ z_2[z_3 + \alpha_2(z_1, z_2, \hat{\theta}) + \varphi_2^{\mathrm{T}}(x_1, x_2)\hat{\theta} \\ &- D^{\nu}\alpha_1(z_1, \hat{\theta})] \end{aligned}$$

If choose $\alpha_2(z_1, z_2, \hat{\theta}) = -C_2 z_2 - z_1 - \varphi_2^{\mathrm{T}}(x_1, x_2)\hat{\theta} + D^{\nu}\alpha_1(z_1, \hat{\theta})$, $z_3, \tilde{\theta}$ are to be governed to zeros. Thus we have

$$\begin{aligned} D^{\nu} V_2 \le &-\sum_{i=1}^{2} C_i z_i^2 + z_2 z_3 \\ &+ \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{2} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^{n} u\hat{\theta}\right) \end{aligned}$$

*Step 3*. Let $z_4 = x_4 - \alpha_3(z_1, z_2, z_3, \hat{\theta})$, we have

$$\begin{aligned} D^{\nu} z_3 = &z_4 + \alpha_3(z_1, z_2, z_3, \hat{\theta}) + \varphi_3^{\mathrm{T}}(x_1, x_2, x_3)\theta \\ &- D^{\nu}\alpha_2(z_1, z_2, \hat{\theta}) \end{aligned}$$

The third fractional Lyapunov function is chosen as $V_3(z_1, z_2, z_3, \hat{\theta}) = V_2 + \frac{1}{2}z_3^2$, we have

$$\begin{aligned} D^{\nu} V_3 \le &-\sum_{i=1}^{2} C_i z_i^2 + z_2 z_3 \\ &+ \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{2} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^{\nu}\hat{\theta}\right) \\ &+ z_3[z_4 + \alpha_3(z_1, z_2, z_3, \hat{\theta}) \\ &+ \varphi_3^{\mathrm{T}}(x_1, x_2, x_3)\theta - D^{\nu}\alpha_2(z_1, z_2, \hat{\theta})] \\ = &-\sum_{i=1}^{2} C_i z_i^2 + z_2 z_3 \\ &+ \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{3} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^{\nu}\hat{\theta}\right) \\ &+ z_3[z_4 + \alpha_3(z_1, z_2, z_3, \hat{\theta}) \\ &+ \varphi_3^{\mathrm{T}}(x_1, x_2, x_3)\hat{\theta} - D^{\nu}\alpha_2(z_1, z_2, \hat{\theta})] \end{aligned}$$

If choose $\alpha_3(z_1, z_2, z_3, \hat{\theta}) = -C_3 z_3 - z_2 - \varphi_3^{\mathrm{T}}(x_1, x_2, x_3)\hat{\theta} + D^{\nu}\alpha_2(z_1, z_2, \hat{\theta})$, $z_4, \tilde{\theta}$ are to be governed to zeros. Thus we have

$$\begin{aligned} D^{\nu} V_3 \le &-\sum_{i=1}^{3} C_i z_i^2 + z_3 z_4 \\ &+ \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{3} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^{\nu}\hat{\theta}\right) \end{aligned}$$

*Step n − 1*. Let $z_n = x_n - \alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta})$, we have

$$\begin{aligned} D^{\nu} z_{n-1} = &z_n + \alpha_{n-1}(z_1, \ldots, z_(n-1), \hat{\theta}) \\ &+ \varphi_{n-1}^{\mathrm{T}}(x_1, \ldots, x_{n-1})\theta - D^{\nu}\alpha_{n-2}(z_1, \ldots, z_{n-2}, \tilde{\theta}) \end{aligned}$$

Let the $(n-1)$th fractional Lyapunov function $V_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}) = V_{n-2} + \frac{1}{2}z_{n-1}^2$, we have

$$\begin{aligned} D^{\nu} V_{n-1} \le &-\sum_{i=1}^{n-2} C_i z_i^2 + z_{n-2} z_{n-1} \\ &+ \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{n-2} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^{\nu}\hat{\theta}\right) \\ &+ z_{n-1}[z_n + \alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}) \\ &+ \varphi_{n-1}^{\mathrm{T}}(x_1, \ldots, x_{n-1})\theta \\ &- D^{\nu}\alpha_{n-2}(z_1, \ldots, z_{n-2}, \hat{\theta})] \\ = &-\sum_{i=1}^{n-2} C_i z_i^2 + + z_{n-2} z_{n-1} \\ &+ \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{n-1} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^{\nu}\hat{\theta}\right) \\ &+ z_{n-1}[z_n + \alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}) \\ &+ \varphi_{n-1}^{\mathrm{T}}(x_1, \ldots, x_{n-1})\hat{\theta} \\ &- D^{\nu}\alpha_{n-2}(z_1, \ldots, z_{n-2}, \hat{\theta})] \end{aligned}$$

If choose $\alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta}) = -C_{n-1} z_{n-1} - z_{n-2} - \varphi_{n-1}^{\mathrm{T}}(x_1, \ldots, x_{n-1})\hat{\theta} + D^{\nu}\alpha_{n-2}(z_1, \ldots, z_{n-2}, \hat{\theta})$, $z_n, \tilde{\theta}$ are to be governed to zeros. Thus we have

$$\begin{aligned} D^{\nu} V_{n-1} \le &-\sum_{i=1}^{n-1} C_i z_i^2 + z_{n-1} z_n \\ &+ \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{n-1} \varphi_i(x_1, \ldots, x_i) z_i - \Gamma^{-1} D^{\nu}\hat{\theta}\right) \end{aligned}$$

*Step n*. The last subsystem $D^{\nu} x_n = \beta(x) u + \varphi_n^{\mathrm{T}}(x)\theta$ can be transformed into

$$D^{\nu} z_n = \beta(x) u + \varphi_n^{\mathrm{T}}(x)\theta - D^{\nu}\alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta})$$

Let the acflf $V_a(z_1, \ldots, z_n, \hat{\theta}) = V_{n-1} + \frac{1}{2}z_n^2$, we have

$$\begin{aligned} D^\nu V_a \le & -\sum_{i=1}^{n-1} C_i z_i^2 + z_{n-1}z_n \\ & + \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{n-1}\varphi_i(x_1, \ldots, x_i)z_i - \Gamma^{-1}D^\nu\hat{\theta}\right) \\ & + z_n[\beta(x)u + \varphi_n^{\mathrm{T}}(x)\theta - D^\nu\alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta})] \\ = & -\sum_{i=1}^{n-1} C_i z_i^2 + z_{n-1}z_n \\ & + \tilde{\theta}^{\mathrm{T}}\left(\sum_{i=1}^{n}\varphi_i(x_1, \ldots, x_i)z_i - \Gamma^{-1}D^\nu\hat{\theta}\right) \\ & + z_n[\beta(x)u + \varphi_n^{\mathrm{T}}(x)\hat{\theta} - D^\nu\alpha_{n-1}(z_1, \ldots, z_{n-1}, \hat{\theta})] \end{aligned}$$

One control and the adaptive law can be chosen by (21) and (22), respectively. So we have $D^\nu V_a \le -\sum_{i=1}^n C_i z_i^2$.

According to Lemma 2, the closed-loop system is stable in the sense of classical Lyapunov stability. Furthermore, we consider two cases (i) and (ii):

(i) When $z \neq 0$, we know $D^\nu V_a < 0$. There exists a $K$-class function $\gamma_1$ such that $D^\nu V_a \le -\gamma_1(\|\bar{z}\|), \bar{z} = [z_1, \ldots, z_n, \tilde{\theta}^{\mathrm{T}}]^{\mathrm{T}}$;

(ii) When $z = 0$, we know $D^\nu V_a \le 0$. If $D^\nu V_a < 0$, similar to case (i), there exists a $K$-class function $\gamma_2$ such that $D^\nu V_a \le -\gamma_2(\|\bar{z}\|)$. However, for the case $D^\nu V_a = 0$, we know $D^\nu V_a = D^\nu C \Longrightarrow V_a = C$, where $C = V_a(t = 0)$ is a positive constant.

If $C = 0$, we know $\tilde{\theta} = 0$ and there exists a $K$-class function $\gamma_3$ such that $D^\nu V_a \le -\gamma_3(\|\bar{z}\|)$. If $C > 0$, we know $\|\tilde{\theta}\| = C'$, where $C'$ is a positive constant only related to $C$ and $\Gamma$.

With respect to Theorem 1, for the case (i), the closed-loop system is asymptotically Mittag–Leffler stable on the region $\Lambda$. Furthermore, on $\mathbb{R}^{m+n}\backslash\Lambda$, when $V_a(t = 0) = 0$, the parameter estimates are asymptotically Mittag–Leffler stable; otherwise, they are bounded by $\{\theta | \|\tilde{\theta}\| = C'\}$.

Therefore the acflf (20) holds on $\Lambda$. So far, this proof is completed. □

Theorems 4 and 5 show how to cope with unknown parameters in fractional-order backstepping, which may result in adaptive fractional-order feedback control laws.

*Remark 6 (Conservative of our main results):* It should be noted that there exist some limitations in Assumptions 1 and 2 to solve Mittag–Leffler stabilisation problems. Technically, the limitations may come from two aspects (i) and (ii). (i) Assumptions 1 and 2 are all based on Theorem 1, which only tells us the sufficient conditions of Mittag–Leffler stability. (ii) These assumptions are derived from simple cflfs provided in Remarks 4 and 5. It is undeniable that there may exist other better cflfs. Therefore our main results of Section 3 are sufficient to stabilise FONSs with a guaranteed Mittag–Leffler stability.

## 4 Simulation experiments

Some examples are presented to illustrate the effectiveness and applicability of the proposed fractional-order backstepping design and to verify the theoretical results. The Grünwald–Letnikov fractional-order difference [2] is used to simulate the FONSs. In our simulations, we abandon the short memory principle for improving numerical accuracy. The time step is set to $h = 0.0001$.

To continue Example 1, another example is considered by numerical simulation as follows.

*Example 3:* Consider a fractional-order non-linear planer system

$$\begin{cases} D^\nu x = \cos(x) - x^3 + \xi \\ D^\nu \xi = u \end{cases} \tag{23}$$

where the fractional order is $\nu = 0.6$ and the equilibrium is $(x, \xi) = (0, -1)$

*Step 1*. Let $z_1 = x, z_2 = \xi - \alpha$, the first fractional Lyapunov function $V_1 = \frac{1}{2}z_1^2$, we have $D^\nu V_1 \le z_1[\cos(x) - x^3 + z_2 + \alpha]$. If choose $\alpha = -\cos(x) + x^3 - C_1 z_1, C_1 > 0$, $z_2$ is to be governed to zero.

*Step 2*. With $D^\nu z_2 = u - D^\nu\alpha$, the cflf $V_2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$, we have $D^\nu V_2 \le -C_1 z_1^2 + z_1 z_2 + z_2(u - D^\nu\alpha)$.

The control can be chosen by $u = -C_2 z_2 - z_1 + D^\nu\alpha, C_2 > 0$.

Finally, the closed-loop system can be written by

$$\begin{cases} D^\nu z_1 = -C_1 z_1 + z_2 \\ D^\nu z_2 = -z_1 - C_2 z_2 \end{cases} \tag{24}$$

In the simulation, $C_1 = C_2 = 1$. The initial pseudo-state is $(6, 5)$. The pseudo-state trajectories of the controlled FONS are shown in Fig. 1. By applying the control, the closed-loop becomes an FOLS. It is seen that the system converges to the equilibrium in a finite time. The control input is shown in Fig. 2.

Together with Example 1, these two examples confirm the appropriateness of the simple cflf for FONSs. The efficiency of the proposed fractional-order backstepping design is demonstrated in solving the FONSs Mittag–Leffler stabilisation problems.

The usefulness and applicability of the proposed method are validated via fractional-order gyroscope system stabilisation in the next example. The gyroscope is a widely used dynamical system and its fractional-order non-linear model attracts some recent attentions [12].

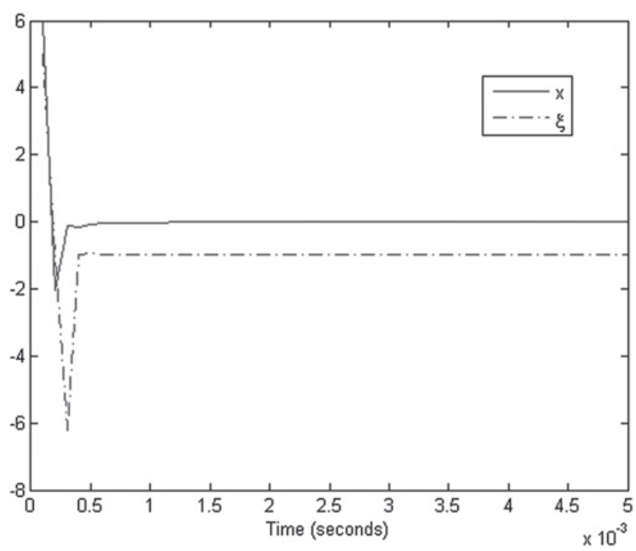


**Fig. 1** *Pseudo-states of Example 3*



doi: 10.1049/iet-cta.2014.0642

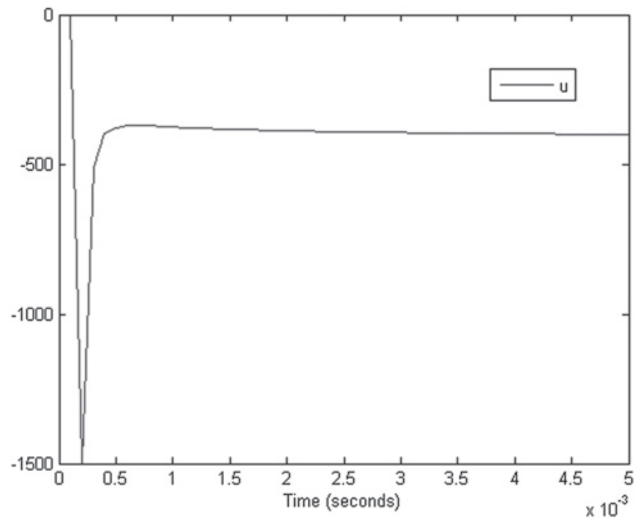


**Fig. 2** *Control input of Example 3*

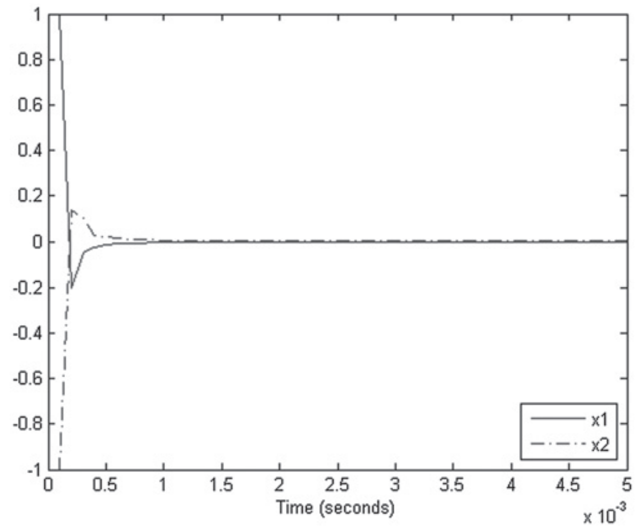


**Fig. 3** *Pseudo-states of Example 4*

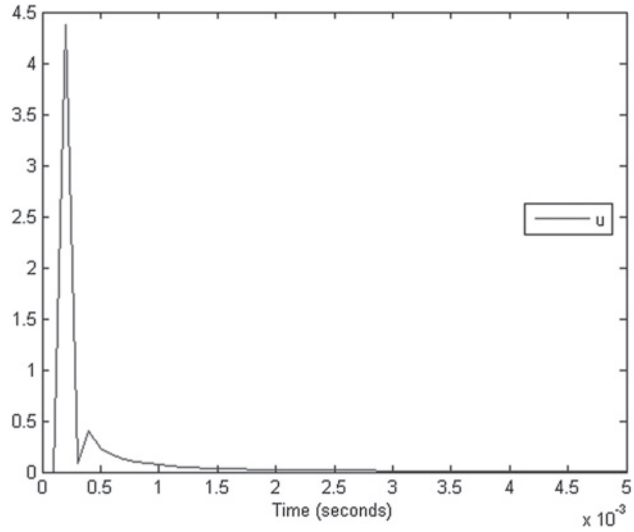


**Fig. 4** *Control input of Example 4*

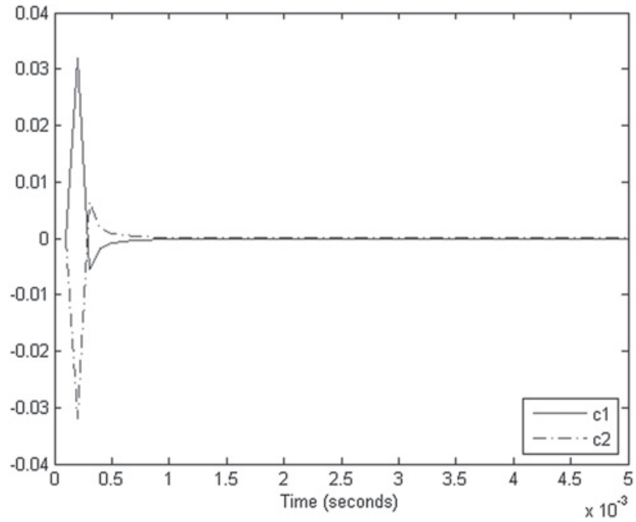


**Fig. 5** *Parameter estimates of Example 4*

*Example 4:* Consider the fractional-order gyroscope system

$$\begin{cases} D^{\nu}x_1 = x_2 \\ D^{\nu}x_2 = -p(t)x_1 - c_1x_2 - c_2x_2^3 + q(t)x_1^3 + u \end{cases} \tag{25}$$

where the fractional order is $\nu = 0.6, p(t) = \frac{\kappa^2}{4} - \beta - f\sin(\omega t), q(t) = \frac{\kappa^2}{12} - \frac{\beta}{6} - \frac{f\sin(\omega t)}{6}$, $\kappa^2 = 100$, $\beta = 1$, $\omega = 25, f = 35.5$. $c_1, c_2$ are viewed as unknown constants, which may be caused by modelling uncertainties.

*Step 1*. Let $z_1 = x_1$, view $x_2$ as the virtual control and $z_2 = x_2 - \alpha_1$, we have $D^{\nu}z_1 = z_2 + \alpha_1(z_1, \hat{c}_1, \hat{c}_2)$.

Note $\tilde{c}_1 = c_1 - \hat{c}, \tilde{c}_2 = c_2 - \hat{c}_2$. Let first fractional Lyapunov function $V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}\tilde{c}_1^2 + \frac{1}{2\rho}\tilde{c}_2^2$. If choose $_1(z_1, \tilde{c}_1, \tilde{c}_2) = -K_1x_1, K_1 > 0$, we have

$$D^{\nu}V_1 \le -K_1z_1^2 + z_1z_2 - \frac{1}{\gamma}\tilde{c}_1D^{\nu}\hat{c}_1 - \frac{1}{\rho}\tilde{c}_2D^{\nu}\hat{c}_2$$

*Step 2*. With $D^{\nu}z_2 = -p(t)x_1 - c_1x_2 - c_2x_2^3 + q(t)x_1^3 + u - D^{\nu}\alpha_1$, let the candidate acflf $V_a = V_1 + \frac{1}{2}z_2^2$.

If choose an adaptive control law

$$\begin{aligned} u &= -K_2(x_2 + K_1x_1) + p(t)x_1 + \hat{c}_1x_2 + \hat{c}_2x_2^3 \\ &\quad - q(t)x_1^3 - K_1x_2 - x_1, \quad K_2 > 0 \\ D^{\nu}\hat{c}_1 &= -\gamma x_2z_2, D^{\nu}\hat{c}_2 = -\rho x_2^3z_2 \end{aligned}$$

Hence, we have $D^{\nu}V_2 \le -K_1z_1^2 - K_2z_2^2$.

In the simulation, $K_1 = 9, K_2 = 6, \rho = \gamma = 1$. The initial pseudo-state is $(1, -1)$ and the initial parameter estimate is $(0, 0)$. The unknown parameters are set to $c_1 = 0.5, c_2 = 0.05$. The pseudo-state trajectories of the controlled system are shown in Fig. 3. By applying the adaptive control, it is seen that the system converges in a finite time. The control input is shown in Fig. 4. The parameter estimates are shown in Fig. 5. It is seen that the proposed fractional-order backstepping is feasible for the adaptive Mittag–Leffler stabilisation of real FOSs.

## 5 Conclusions

In this paper, it may be the first time to investigate commensurate FONSs Mittag–Leffler stabilisation. With respect to the power law for fractional-order derivative, several conditions of control fractional Lyapunov functions (cflfs) are presented. Fractional-order backstepping is developed to stabilise DFONSs and UFONSs. The analytical forms of two kind control laws are presented. Finally, the effectiveness and applicability of the proposed technique are verified.

The future study topics can be directed to the application of the proposed fractional-order backstepping in various FOSs, such as triangular FONSs [20], networked FONSs [36, 37] and fractional-order PMSM [38], etc.

## 6 Acknowledgments

The authors are grateful to anonymous reviewers' comments and Dr. Peng for constructive talks on the proofs. This work was supported by the National Natural Science Foundation of China under Grant no. 61171034, the Fundamental Research Funds for the Central Universities and the Province Natural Science Fund of Zhejiang under Grant no. R1110443.

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doi: 10.1049/iet-cta.2014.0642